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Trembling Invisible Hand Equilibrium

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This paper shows how to find approximate strong Markov equilibria in a setting of infinite horizon incomplete markets by solving a finite dimensional fixed point problem. These equilibria have the interpretation of a slightly imperfect and slightly random market mechanism. *Journal of Economic Literature* Classification Numbers: C62, D52, D92, G10.

1. INTRODUCTION

A strong Markov equilibrium describes an economic system with an indefinite past and indefinite future. We need not imagine that this equilibrium truly has existed forever in the past, that it will do so in the future, nor yet that the agents believe that it has or will. Rather, if the process has gone for a long time, and is stationary and ergodic, traders will have drawn accurate inferences about it. Since the process is expected to continue for a long time, traders will use these inferred relations in making forecasts. In other words, in a strong Markov equilibrium rational expectations make sense. Note, however, that there is nothing persuasive about the requirement that traders have exact knowledge of the relations involved.

In rational expectation models where equilibria solve a social welfare problem, tools from dynamic optimization theory can be applied to prove a stationary ergodic equilibrium exists: see Marimon [7], for example. Indeed, strong Markov equilibria are the basis of modern recursive analysis as exposited, for example, in Stokey *et al.* [9]. With overlapping generations, or market imperfections it is not currently known whether strong Markov equilibria exist under reasonably general conditions, although they are known to exist for certain special cases (see Levine [6]). In a different

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context, Spear [8] has shown that they generically fail to exist if we require the invariant distribution to have finite support. It is known from the work of Duffie *et al.* [2], that a weaker form of stationary equilibria with some ergodic properties exist, and Kehoe and Levine [4] give a method of computing them. But the high dimensionality of the state space, computational and analytic intractability, and the need to include economically irrelevant variables make these equilibria less appealing than strong Markov equilibria.

On the other hand, strong Markov equilibrium supposes that traders have exact knowledge of the causal links between initial conditions and equilibrium outcomes. It is more sensible to assume that their knowledge of this causal link is fuzzy. There are two modeling approaches: to assume that the world is exact, but knowledge fuzzy, or to assume that knowledge is fuzzy because the world is fuzzy. We take the latter approach, because it leads to a tighter model.

To be concrete, we focus on an economy where intertemporal and interstate trade may be carried out only by trading ownership in a finite collection of long-lived assets which may not be held short. In particular, markets are potentially incomplete. This may be interpreted as a commodity money variant of Bewley [1] and Townsend [10]. Indeed, if the assets represent physically transportable goods that yield a flow of services, this type of equilibrium follows immediately from the Townsend turnpike spatial location model. The incomplete market/no short sales model has also been studied by Duffie *et al.* [2], Levine [5, 6], and others.

Since we cannot prove the existence of a strong Markov equilibrium in this setting, we weaken the notion of equilibrium. Ordinarily, we describe the market mechanism as choosing a price, and assume that individuals can buy or sell as much as they like at that price. Of course, if the market chooses the wrong price the resulting allocation will not be socially feasible. Consequently, it is better to think of the market as choosing both a price and allocation so that no trader is frustrated by the allocation he receives. Recognizing that the "market" is a complex and nonmodeled process, and that competitive equilibrium is merely an idealization, we relax this to require only that the market chooses a price and allocation so that no trader is very frustrated by the allocation he receives. In addition, we want to allow market outcomes to be imperfectly predictable, due to small nonmodeled factors. So we assume that the market chooses a price and a random allocation in such a way that traders are not very frustrated in expected value. This does allow the possibility that the market selects allocations that lead to a very high degree of frustration, but it must do so with very low probability. As a result there is little incentive to try to improve the market mechanism: the market gets things almost right most of the time. This theory is consistent with the existence of a small amount of monopoly power, asymmetric information, imperfect computation, and other frictions.

We refer to such a mechanism as a trembling invisible hand equilibrium: the invisible hand of the market is slightly imperfect in simultaneously satisfying all agents. We are then able to give a concrete model of such equilibria as exact equilibria of an economy with small random income transfers. These equilibria can be calculated by solving a finite fixed point problem, much as in the nonstochastic analysis of steady states, and are an approximate stochastic steady state.

2. The Model

We study an intertemporal exchange economy in which there are many assets, each representing ownership in a stream of consumption goods. Debts cannot be collected, so borrowing and lending can only take place through the sale and purchase of assets.

At each time $t = ..., -1, 0, 1, ..., \eta_t$ is an index of current and future technological possibilities for the economy. The exogenous state variable η_t lies in a finite set $\eta_t = 1, 2, ..., I$, and forms a Markov chain with transition chain probabilities denoted $\pi(\eta_{t+1} \mid \eta_t)$. These are assumed to be strictly positive:

$$\pi(\eta_{t+1} \mid \eta_t) > 0. \tag{A.1}$$

At the beginning of period t, the current and past values $\eta_t, \eta_{t-1}, n_{t-2}, ...,$ are assumed to be commonly known and the future values $\eta_{t+1}, \eta_{t+2}, ...,$ are unknown.

There are finitely many traders $a = 1, 2, ..., A$. There are C consumption goods, and at time t trader a consumes $x_t^a \in \mathbb{R}_+^C$. Endowments are a function of the exogenous state, denoted $\bar{x}^a(\eta_t) \in \mathbb{R}_+^C$. We assume

$$\bar{x}^a(\eta_t) > 0 \tag{A.2}$$

At any time t, trader a has utility depending on current and future consumption. This has the intertemporally separable form

$$U_t^a = \mathrm{E}_t\left[\sum_{\tau=t}^{\infty} \delta^{t-\tau} u^a(x_\tau^a, \eta_\tau)\right], \tag{2.1}$$

where $0 \leqslant \delta < 1$ is a fixed common subjective discount factor, E_t is the expectation conditional on information dated t and earlier, and the period utility function u^a satisfies:

> $u^a(x_t^a, \eta_t)$ is bounded, and as a function of x_t^a on $\mathbb{R}_+^C$, it is strictly increasing, concave, and smooth. The gradient $\mathrm{D}u^a(x_t^a, \eta_t)$ is bounded even on the boundary of the consumption set. (A.3)

The assumption that $Du^a(x_t^a, n_t)$ is bounded on the boundary of the consumption set is not standard (although not unreasonable). It is used to guarantee that marginal utility is bounded across all equilibria. In this setting of no short sales, it is a technical artifice in the following sense: since endowments are strictly positive and no borrowing is possible, consumption is uniformly bounded away from zero in every equilibrium and so is marginal utility, even if it becomes infinite on the zero consumption boundary. However, making this argument formal adds a significant amount of notation without adding insight. Those interested in the details of this type of argument will find it carefully laid out in Duffie *et al.* [2].

There are N assets. At time t, the beginning of period asset holdings of a type a trader are denoted by $y_t^a \in \mathbb{R}_+^N$. We will also use z's for the end of period holdings, and omit the superscript a as y_t, for the vector of all traders' asset holdings. Notice that asset holdings must be nonnegative, to reflect the fact that debts cannot be collected. Each asset j held at the end of period $t-1$ yields a consumption return of $R^j(\eta_t) \in \mathbb{R}^C$ at the beginning of period t. Consistent with limited liability, returns must be nonnegative:

$$R^j(\eta_t) \geqslant (\neq)\, 0. \tag{A.4}$$

Letting $R(\eta_t)$ denote the matrix with columns $R^j(\eta_t)$, a trader who holds y_t^a at the beginning of period t receives goods $R(\eta_t)\, y_t^a$ then.

There is a fixed positive stock of assets $\bar{y} \in \mathbb{R}_{++}^N$. Social feasibility therefore requires that

$$\sum_{a=1}^{A} y_t^a = \bar{y} \tag{E.1}$$

in each and every period. The total available consumption consists of aggregate individual endowments, $\bar{x}(\eta_t) \equiv \sum_{a=1}^{A} \bar{x}^a(\eta_t)$ which, due to the impossibility of debt collection, can be traded only on spot markets, plus the consumption for which ownership is transferable through asset trading $R(\eta_t)\, \bar{y}$. Social feasibility of consumption, then, requires that

$$\sum_{a=1}^{A} x_t^a = \bar{x}(\eta_t) + R(\eta_t)\, \bar{y}. \tag{E.2}$$

Let p_t denote the prices of consumption goods, and q_t of assets. By the monotonicity of preferences, and nonnegativity of asset returns, we may assume prices are nonnegative. A trader of type a who held a portfolio of y_t^a at the beginning of period t, faces a budget constraint of

$$p_t(x_t^a - \bar{x}^a(\eta_t) - R(\eta_t)\, y_t^a) + q_t(y_{t+1}^a - y_t^a) \leqslant 0. \tag{E.3}$$

Because this is homogeneous in prices, we are free to normalize prices so that the vector (p_t, q_t) is on the unit simplex.

A *strong Markov plan* for this economy consists of functions $p(y_t, \eta_t)$, $q(y_t, \eta_t)$, $x(y_t, \eta_t)$, and $z(y_t, \eta_t)$ together with a stationary ergodic process p_t, q_t, x_t, y_t, η_t that almost surely satisfies $p_t = p(y_t, \eta_t)$, $q_t = q(y_t, \eta_t)$, $x_t = x(y_t, \eta_t)$, $y_{t+1} = z(y_t, \eta_t)$, and

$$\operatorname{Prob}(\eta_{t+1} \mid \eta_t, \eta_{t-1}, ..., y_t, y_{t-1}, ...) = \pi(\eta_{t+1} \mid \eta_t).$$

We also consider the weaker notion of a *random Markov plan*. Here we replace $z(y_t, \eta_t)$ with conditional probability distributions $\pi(y_{t+1}, \eta_{t+1} \mid y_t, \eta_t)$, and rather than requiring $y_{t+1} = z(y_t, \eta_t)$, we require that

$$\operatorname{Prob}(y_{t+1}, \eta_{t+1} \mid \eta_t, \eta_{t-1}, ..., y_t, y_{t-1}, ...) = \pi(y_{t+1}, \eta_{t+1} \mid y_t, \eta_t).$$

A strong Markov Plan is a *strong Markov equilibrium* if the social feasibility conditions (E.1) and (E.2) are almost surely satisfied by x_t and y_t. Moreover, it must be true that for each time τ and agent a, and almost all initial portfolios y_τ, the plan $x^a(y_t, \eta_t)$, $z^a(y_t, \eta_t)$ maximizes the objective function (2.1) subject to the budget constraints (E.3) for $t \geqslant \tau$.

3. Trembling Invisible Hand Equilibrium

Our goal is to look for equilibria in which there is not an exact link between initial conditions and outcomes. To do this we introduce an imperfect market mechanism in which portfolios at the beginning of a period are a stochastic function of portfolios at the end of the previous period.

Informally, an ε *trembling invisible hand equilibrium* is a random Markov plan such that the social feasibility conditions (E.1) and (E.2) are almost surely satisfied by x_t and y_t. Moreover, it must be true that for each τ and agent a, and almost all initial portfolios y_τ, the utility in (2.1) from the allocation $x^a(y_t, \eta_t)$ cannot be improved upon by more than ε by any plan satisfying the budget constraints (E.3) for $t \geqslant \tau$.

Notice that in the definition of a random Markov plan, the randomness in the market allocation at t is assumed to lie entirely in the portfolio y_{t+1} held at the beginning of $t+1$. This creates fuzziness in the causal link between time t and $t+1$: at t traders cannot be exactly certain what the relationship between η_{t+1} and p_{t+1} will be.

To give a precise mathematical definition of an ε-trembling invisible hand equilibrium, fix a random Markov plan. Let

$$v^a(p_t, I_t^a, \eta_t) = \max_{x_t^a} u^a(x_t^a, \eta_t) \qquad \text{subject to} \quad p_t x_t^a \leqslant I_t^a + p_t \bar{x}^a(\eta_t). \tag{3.1}$$

Also define

$$r_t \equiv p_t R(\eta_t) + q_t \tag{3.2}$$

to be the total return on each asset.

Now consider a price taking agent who deviates from the random Markov plan by choosing the portfolio process $\tilde{y}_t^a$. The unique dynamic programming value function satisfies

$$\begin{aligned} &V^a(y_t, \tilde{y}_t^a, \eta_t) \\ &\quad = \max_{I_t^a, \tilde{y}_{t+1}^a} v^a(p_t, I_t^a, \eta_t) + \delta \int V^a(y_{t+1}, \tilde{y}_{t+1}^a, \eta_{t+1})\, d\pi(y_{t+1}, \eta_{t+1} \mid y_t, \eta_t) \\ &\qquad \text{subject to} \quad I_t^a = r_t \tilde{y}_t^a - q_t \tilde{y}_{t+1}^a \geqslant -p_t \bar{x}^a(\eta_t). \end{aligned} \tag{3.3}$$

This represents the most utility an agent starting with a $\tilde{y}_t^a$ can achieve if allowed to trade freely. (Note that y_t appears in the value function because it is the state variable for the Markov process followed by prices, and so forth.) Let

$$W^a(y_t, \eta_t) = \mathrm{E}\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} u^a(x_\tau^a, \eta_\tau) \mid y_t, \eta_t\right] \tag{3.4}$$

be the realization of the conditionally expected present value at time t according to the proposed equilibrium. Then to be an ε-trembling invisible hand equilibrium we require that for all agents a

$$\int [V^a(y_t, y_t^a, \eta_t) - W^a(y_t, \eta_t)]\, \mathrm{d}\pi(y_t, \eta_t) \leqslant \varepsilon, \tag{E.4}$$

where $\pi(y_t, \eta_t)$ are the stationary probabilities. In other words, the loss relative to the optimum, weighted by the probability with which events occur, should not exceed ε. This allows for the possibility that there is an improbable value of (y_t, η_t) for which the loss is very large.

Our goal in this paper is to prove:

THEOREM 3.1. *Under* (A.1) *to* (A.4), *ε-trembling invisible hand equilibria exist for all positive values of* ε.

Proof. Follows from Theorems 3.2 and 3.3 below. ▮

We will show that not only do such equilibria exist, but that they exist with finite support: that is, the measure $\pi(y_{t+1}, \eta_{t+1} \mid y_t, \eta_t)$ puts all its weight on a finite set of portfolios y_{t+1}, fixed independent of y_t and η_t. Consequently, we may let S denote the finite set of pairs (y_t, η_t) that have positive weight. We may then write $p_s, q_s, x_s, \pi_{s\sigma}$ in place of $p(y_t, \eta_t)$, $q(y_t, \eta_t)$, $x(y_t, \eta_t)$, and $\pi(y_{t+1}, \eta_{t+1} \mid y_t, \eta_t)$ where $s = (y_t, \eta_t)$ and $\sigma = (y_{t+1}, \eta_{t+1})$. Equivalently, we may define a *finite Markov plan* by specifying a finite set S, and values y_s, η_s showing how initial portfolio state pairs depend on s, plus values p_s, q_s, x_s, and $\pi_{s\sigma}$. The requirement of consistency with $\pi(\eta_{t+1} \mid \eta_t)$ is that if $s(\eta)$ are all the states s with $\eta_s = \eta$, then $\sum_{\sigma \in s(\eta)} \pi_{s\sigma} = \pi(\eta \mid \eta_s)$. We also require that the finite Markov chain S, $\pi_{s\sigma}$ have a single ergodic class and let π_s denote the unique stationary probabilities. If $s_t = s$, we put $p_t = p_s$, $q_t = q_s$, $x_t = x_s$, $y_t = y_s$, and $\eta_t = \eta_s$. In this way, each finite Markov plan gives rise to a random Markov plan.

We now consider a fictitious finite economy defined on S. Our goal will be to show that equilibria of this economy give rise to finite Markov plans that are ε-trembling invisible hand equilibria. First, fix a finite set S, and associate each point $s \in S$ with an exogenous state η_s.

In the fictitious economy, each state $s \in S$ is associated with a market: agent a consumes x_s^a, trades at p_s, and is endowed with $\bar{x}^a(\eta_s)$. Associated with each market is a strictly positive probability weight π_s; agent a's objective is

$$\sum_{s \in S} \pi_s u^a(x_s^a, \eta_s) \tag{3.5}$$

that is, he seeks to maximize the stationary value of his utility.

Our goal is to define budget constraints that link the markets through portfolio holdings in such a way that an agent has the same incentive for holding securities as in the dynamic economy. To do this, we associate with each state and agent a a portfolio z_s^a that is to be interpreted as a target portfolio he "should hold" at the end of market s. These portfolios should be socially feasible, so that $\sum_{a=1}^{A} z_s^a = \bar{y}$. We also associate with each pair of states a probability $\mu_{s\sigma}$ with $\sum_{s \in S} \mu_{s\sigma} = 1$. This is interpreted as the probability at σ of having previously been at s. Let q_s denote the price of purchasing securities at s, $\tilde{z}_s^a$ agent a's chosen end of period securities holdings, and define $r_s = p_s R_s(\eta_s) + q_s$. Then in market s, agent a is constrained by nonnegative consumption and securities holdings, and the budget constraint

$$p_s(x_s^a - \bar{x}^a(\eta_s)) + q_s \tilde{z}_s^a - r_s \sum_{\sigma \in S} \mu_{\sigma s}[(1 - \delta - \gamma) z_\sigma^a + \delta \tilde{z}_\sigma^a + \gamma \bar{y}/A] \leqslant 0, \tag{3.6}$$

where $0 < \gamma < 1 - \delta$. In other words, at s, agent a's initial portfolio of assets is a weighted average of his holdings at the end of different states σ, each of which in turn is a weighted average of what his holdings "should" be, and what he actually chose to hold, with the weight on what he chose equal to his own discount factor. In addition, to guarantee positive endowments, each agent is given a small fraction γ/A of available assets. This is a technical artifice that simplifies proofs.

An *equilibrium of the fictitious econmy* comprises prices p_s, q_s, and plans x_s and $\tilde{z}_s$ that are socially feasible and individually optimal. In addition, we require that the target portfolio z_s satisfies

$$z_s^a = \tilde{z}_s^a, \tag{3.7}$$

so that each agent chooses to hold the portfolio he "should" hold.

In equilibrium, by (3.7), each agent's beginning of period asset portfolio at s is

$$y_s^a = (1-\gamma) \sum_\sigma \mu_{\sigma s} z_\sigma^a + \gamma \bar{y}/A. \tag{3.8}$$

Moreover, we may define $\pi_{s\sigma}$ by

$$\pi_{s\sigma} = \frac{\pi_\sigma \mu_{s\sigma}}{\pi_s}. \tag{3.9}$$

This implies that the $\mu_{s\sigma}$ are the stationary probabilities of having last period been at s, given the system is now at σ.

Each equilibrium of the fictitious economy gives rise, therefore, to a finite Markov plan y_s, η_s, p_s, q_s, x_s, and $\pi_{s\sigma}$ provided that $\sum_{\sigma \in s(\eta)} \pi_{s\sigma} = \pi(\eta \mid \eta_s)$. Rewriting the latter condition in terms of the $\mu_{s\sigma}$, we say the fictitious economy is *consistent* if $\sum_{\sigma \in s(\eta)} \pi_\sigma \mu_{s\sigma} = \pi_s \pi(\eta \mid \eta_s)$. Since the Markov chain $\pi_{s\sigma}$ has a unique ergodic class if and only if the time reversed Markov chain $\mu_{s\sigma}$ does, we call the fictitious economy *ergodic* in the latter case. Our conclusion is that an equilibrium of a consistent ergodic fictitious economy gives rise to a finite Markov plan.

We may now compute the difference between the value of the fixed portfolio from (3.8), y_s^a, held at the beginning of s in the fictitious economy, and the actual portfolio held at the end of σ in the corresponding finite Markov plan, z_σ^a. Remember that in a random Markov plan, these are not required to be equal. Define

$$\Delta_{\sigma s}^a = r_\sigma (y_s^a - z_\sigma^a). \tag{3.10}$$

Define also net asset income

$$J_s^a = r_s y_s^a - q_s z_s^a. \tag{3.11}$$

Finally, let the Lagrange multiplier associated with the constraint (3.6) in state s be $\pi_s \phi_s^a$, and compute the marginal utility of zero income as

$$\lambda_s^a = \begin{cases} \phi_s^a & J_s^a = -p_s \bar{x}^a(\eta_s) \\ D_I v^a(p_s, -p_s \bar{x}^a(\eta_s)) & J_s^a > -p_s \bar{x}^a(\eta_s). \end{cases} \tag{3.12}$$

Our main theorem, Theorem 3.1, follows from the following two results.

THEOREM 3.2. *Under* (A.1) *to* (A.4) *a finite Markov plan arising as an equilibrium of a consistent ergodic fictitious economy is an ε-trembling invisible hand equilibrium with*

$$\varepsilon = (1-\delta)^{-1}\{\max_{\{a,s,\sigma \mid \mu_{s\sigma}>0\}} \Delta^a_{s\sigma}\}\left\{\max_a \sum_s \pi_s \lambda^a_s\right\}.$$

Proof. Follows from Theorems 4.1 and 4.2. ∎

Theorem 3.1 then follows from a theorem on fictitious economies.

THEOREM 3.3. *Under* (A.1) *to* (A.4), *there exists a* $B>0$ *such that for all* $\varepsilon>0$ *there is a set S and a consistent ergodic equilibrium of a fictitious economy on S satisfying*

$$\sum_{s\in S} \lambda^a_s \pi_s \leqslant B \tag{3.13}$$

$$(\Delta^a_{s\sigma} =)\ |r_s(y^a_\sigma - z^a_s)| \leqslant \varepsilon \qquad \textit{whenever} \quad \mu_{s\sigma}>0. \tag{3.14}$$

Proof. See Section 5. ∎

4. TRANSFER PAYMENT EQUILIBRIA

In this section we prove Theorem 3.2, that each equilibrium of a consistent ergodic fictitious economy gives rise to a finite Markov plan that is an ε-trembling invisible hand equilibrium. Where previously we examined approximate equilibria of the exact model, we now turn to exact equilibria of an approximate model. The approximation involves modifying budget constraints to include small random transfer payments. These transfer payment equilibria do not have as good an economic interpretation as ε-trembling invisible hand equilibria. On the other hand, they are easier to work with, and we show that if the transfer payments are small, so is the ε for the corresponding ε-trembling invisible hand equilibrium. The advantage of transfer payment equilibria is that they are the same as equilibria of fictitious economies. This gives us a method of calculating ε-trembling invisible hand equilibria.

For given prices p_t, and state η_t, suppose that λ^a_t is any scalar satisfying

$$\lambda^a_t \geqslant D_I v^a(p_t, -p_t\bar{x}^a(\eta_t), \eta_t), \tag{4.1}$$

and for $I^a_t \leqslant -p_t\bar{x}^a(\eta_t)$ define

$$v^a(p_t, I^a_t, \eta_t, \lambda^a_t) \equiv v^a(p_t, -p_t\bar{x}(\eta_t), \eta_t) + \lambda^a_t(I^a_t + p_t\bar{x}^a(\eta_t)). \tag{4.2}$$

In other words, we use λ_t^a to extend the indirect period utility functions to negative levels of total income. Moreover, (4.1) ensures that v^a is concave in income. The reason for extending v^a to negative income levels is that random income transfers will generally cause involuntary bankruptcy in some states.

An *ε-transfer payment equilibrium* is a finite Markov plan plus multipliers λ_s and income transfers $\Delta_{s\sigma}$, that satisfy certain properties. We interpret $\Delta_{s\sigma}^a$ to be the additional income transferred to a in state σ when the previous state was s. We require

$$|\Delta_{s\sigma}^a| \leqslant \varepsilon, \tag{4.3}$$

that is, the income transfers should not exceed ε. Second, for $s=(y_t, \eta_t)$, $\sigma=(y_{t+1}, \eta_{t+1})$ define

$$\Delta_{t+1}=\Delta_{s\sigma}; \qquad \lambda_t=\lambda_s.$$

Then there must exist a sequence of $\tilde{y}_t^a$ that solve the optimization problem

$$\max \mathrm{E}\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} v^a(p_\tau, J_\tau^a, \eta_\tau, \lambda_\tau^a) \,\middle|\, y_t, \eta_t\right]$$

$$\text{subject to} \quad J_\tau^a = r_\tau \tilde{y}_\tau^a - q_\tau \tilde{y}_{\tau+1}^a \begin{cases} +\Delta_\tau^a, & \tau>t \\ +0, & \tau=t \end{cases} \tag{4.4}$$

almost surely for each t. This is the same optimization problem as (3.3), used in defining an ε-trembling invisible hand equilibrium, except that there are random income transfers Δ_τ^a. We also require that the optimal $\tilde{y}_\tau^a$ and the corresponding J_t^a must almost surely satisfy

$$r_\tau(y_\tau^a - \tilde{y}_\tau^a) = \Delta_\tau^a, \tag{4.5}$$

$$J_\tau^a \geqslant -p_\tau^a \bar{x}(\eta_\tau) \tag{4.6}$$

although these conditions are not part of the constraints on the optimization problem (4.4). In other words, the difference between the optimal and equilibrium income exactly equals the income transfer, and the optimal total income is nonnegative. If we substitute (4.5) into the definition of J_τ^a, we find $J_\tau^a = r_\tau y_\tau^a - q_\tau \tilde{y}_{\tau+1}^a$, so that if the optimal plan has been followed through t, the initial condition depends on t only through s_t (that is, y_t), and is independent of $s_{t-1}, s_{t-2}, \ldots$. This means that an optimum depending only on s exists. In particular, at such an optimum $J_\tau = J_s$ depending only on s.

To understand this definition, suppose that S, x_s, p_s, q_s, y_s, $\pi_{s\sigma}$, λ_s, and $\Delta_{s\sigma}$ form a transfer payment equilibrium. It is clear that $\tilde{y}_\tau^a = y_\tau^a$ must be optimal in (4.4) since by (4.5) it yields the same income as the optimal plan. Moreover, from (4.6), and the fact that J^a_t depends only on s_t, it follows that there is a unique x^a_s associated with each state such that $u^a(x^a_s, \eta_s) = v^a(p_s, J^a_s, \eta_s, \lambda^a_s)$. Consequently, the joint savings, consumption plan x_t, y_t is an optimal solution to maximizing (3.4) subject to budget constraints (E.3). In other words together with the uniquely defined consumption plans, a transfer payment equilibrium is a 0-trembling invisible hand equilibrium. For our purposes, we need the following extension of this result:

THEOREM 4.1. *Under* (A.1) *to* (A.4), *every* ε*-transfer payment equilibrium, together with the unique associated consumption plan, is an* ε'*-trembling invisible hand equilibrium with*

$$\varepsilon' = \varepsilon(1-\delta)^{-1} \max_a \sum_{s \in S} \pi_s \lambda^a_s.$$

Remark. This implies the previous result, since in equilibrium it is clear that λ^a_s must be finite. What this theorem shows is that to prove ε-trembling invisible hand equilibria exist for all positive ε, we must show that there is a sequence of ε'-transfer payment equilibria with $\varepsilon' \to 0$ and $\sum_{s \in S} \pi_s \lambda^a_s$ bounded above.

Proof. What we must show is that the random income transfers do not hurt any agent too much. The realized utility in (3.4) is found by solving (4.4); the maximal utility in (3.3) by solving

$$\max \mathrm{E}\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} v^a(p_\tau, I^a_\tau, \eta_\tau, \lambda^a_\tau) \,\middle|\, y_t, \eta_t \right]$$

$$\text{subject to} \quad I^a_\tau = r_\tau y^a_\tau - q_\tau \tilde{y}^a_{\tau+1} \geqslant -p_\tau \bar{x}^a(\eta_\tau), \qquad \tau \geqslant t. \tag{4.7}$$

Suppose $\tilde{y}^a_\tau$ is feasible in (4.7), and let I^a_τ be the corresponding income. Since (4.4) does not require $J^a_\tau \geqslant -p_\tau \bar{x}^a(\eta_\tau)$, the plan $\tilde{y}^a_\tau$ is also feasible there and yields the utility

$$\mathrm{E}\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} v^a(p_\tau, I^a_\tau + \Delta^a_\tau, \eta_\tau, \lambda^a_\tau) \,\middle|\, y_t, \eta_t \right]. \tag{4.8}$$

Moreover, by (4.1) $v^a(p_t, ..., \eta_t, \lambda^a_\tau)$ is Lipshitz in I^a with Lipshitz constant λ^a_t. By (4.3), this implies that (4.7) exceeds (4.8) by at most

$$\varepsilon \mathrm{E}_t \sum_{\tau=t}^{\infty} \delta^{\tau-t} \lambda^a_\tau. \tag{4.9}$$

Taking unconditional expectations with respect to the stationary probabilities then yields the desired result. ∎

A key aspect of an equilibrium with transfer payments is that in equilibrium the transfer payments make the value of an agents endowment depend only on the current state. We can use this observation to construct a fictitous finite economy, such that the first order conditions for an individual optimum are the same as those in an equilibrium with transfer payments. This shows that the finite Markov plan arising from such a fictitious economy may be transformed into an equilibrium with transfer payments by simply defining the transfer payments in such a way as to give each agent the same income he has in the fictitious equilibrium.

Consider in a fictitious economy the first order conditions for an agent maximizing (3.5) subject to (3.6), and recall that the Lagrange multiplier associated with the constraint (3.6) in state s is $\pi_s \phi_s^a$. We get (with complementary slackness)

$$\begin{gathered} \mathrm{D}u^a(x_s^a, \eta_s) \leqslant \phi_s^a p_s \\ \phi_s^a q_s \geqslant \delta \sum_{\sigma \in S} (\pi_\sigma \mu_{s\sigma}/\pi_s)\, \phi_\sigma^a r_\sigma. \end{gathered} \tag{4.10}$$

In terms of the extended indirect utility function, net asset income J_s^a, and the multipliers λ_s^a, this may be written

$$\mathrm{D}_I v^a(p_s, J_s^a, \eta_s, \lambda_s^a) q_s \geqslant \delta \sum_{\sigma \in S} (\pi_\sigma \mu_{s\sigma}/\pi_s)\, \mathrm{D}_I v^a(p_\sigma, J_\sigma^a, \eta_\sigma, \lambda_\sigma^a) r_\sigma. \tag{4.11}$$

This can be contrasted with the first order condition for a transfer payment equilibrium. Note that in such an equilibrium J_s^a depends only on s and we may write

$$\mathrm{D}_I v^a(p_s, J_s^a, \eta_s, \lambda_s^a) q_s \geqslant \delta \sum_{\sigma \in S} \pi_{s\sigma}\, \mathrm{D}_I v^a(p_\sigma, J_\sigma^a, \eta_\sigma, \lambda_s^a) r_\sigma. \tag{4.12}$$

However, by (3.9), defining $\pi_{s\sigma}$, (4.11) and (4.12) are the same, and we can now prove:

THEOREM 4.2. *Under* (A.1) *to* (A.4), *every equilibrium of a consistent ergodic fictitious economy gives rise to a finite Markov plan, that together with the multipliers* (3.12) *and transfers* (3.10) *is a transfer payment equilibrium.*

Proof. By the discussion above, we need only show that the first order condition (4.12) is sufficient for an optimum in (4.4). This follows from the concavity and boundedness of u^a together with the fact that the proposed optimal plan is stationary, and consequently yields finite utility. See Weitzman [11], Bewley [1], or Levine [5]. ∎

5. Existence of Trembling Invisible Hand Equilibrium

From the previous two sections, we have reduced the problem of the existence of ε-trembling invisible hand equilibria to showing the existence of fictitious economy equilibria in which the transfers (3.10) are uniformly small, and the multipliers from (3.12) are bounded in expected value. We do this by associating the state space S with a grid in the space of feasible portfolios, and refining the grid. Recall the statement of

Theorem 3.3. *Under* (A.1) *to* (A.4), *there exists a* $B>0$ *such that for all* $\varepsilon>0$, *for each such* ε *there is a set* S *and a consistent ergodic equilibrium of a fictitious economy on* S *satisfying*

$$\sum_{s\in S} \lambda_s^a \pi_s \leqslant B \tag{5.1}$$

$$|r_s(y_\sigma^a - z_s^a)| \leqslant \varepsilon, \qquad \text{whenever} \quad \mu_{s\sigma}>0. \tag{5.2}$$

Proof. Fix $\alpha=\varepsilon[1+\max_{\eta_t\in I} R(\eta_t)]^{-1}/2$ and γ so that $\gamma=\alpha/2|\bar{y}|$. Let $Y\subset\mathbb{R}_+^{NA}$ be the space of socially feasible portfolios, and, since this space is a compact metric space, let Y^α be a finite subset of Y such that every point in Y is within α of a point in Y^α. Define $S'\equiv Y^\alpha\times I$. For $z_s\in\mathbb{R}_+^{NA}$ representing an end of period portfolio, $y^\alpha\in Y^\alpha$ representing a beginning of the following period portfolio, and n an integer, define

$$d^n(y^\alpha\,|\,z_s)=[|y^\alpha-z_s|+\alpha]^{-n}.$$

The "probability of moving from z_s to y^α" is then defined as

$$\hat{\pi}^n(y^\alpha\,|\,z_s)=\frac{d^n(y^\alpha\,|\,z_s)}{\sum_{y\in Y^\alpha} d^n(y\,|\,z_s)}.$$

Observe that $\hat{\pi}^n(y^\alpha\,|\,z_s)>0$, $\sum_{y\in Y_\alpha}\hat{\pi}^n(y^\alpha\,|\,z_s)=1$, and $\hat{\pi}^n(y^\alpha\,|\,z_s)$ is continuous in z_s. Moreover, if $Y^\alpha(z_s)$ are the set of points in Y^α closest to z_s

$$\lim_{n\to\infty}\sum_{y^\alpha\in Y^\alpha(z_s)}\hat{\pi}^n(y^\alpha\,|\,z_s)\to 1. \tag{5.3}$$

Let $s=(y_s^\alpha,\eta_s)$; $\sigma=(y_\sigma^\alpha,\eta_\sigma)$. If $z\in\mathbb{R}_+^{NAS}$, we define $\pi_{s\sigma}^n(z)=\pi(\eta_\sigma\,|\,\eta_s)\,\hat{\pi}^n(y_\sigma^\alpha\,|\,z)$. Then $\pi_{s\sigma}^n(z)>0$ and $\sum_{\sigma\in s}\pi_{s\sigma}^n(z)=1$, so for each z $\pi_{s\sigma}^n(z)$ defines a Markov process with strictly positive transition probabilities. It follows that there is a unique positive stationary distribution $\pi_s^n(z)$ and a unique reverse Markov process $\mu_{s\sigma}^n(z)$ satisfying (3.9). Moreover, since $\hat{\pi}^n(y\,|\,z_s)$ is continuous in z_s, $\pi_{s\sigma}^n(z)$ is continuous in z, and it follows from the theory of Markov chains that $\pi_s^n(z)$ and $\mu_{s\sigma}^n(z)$ are continuous in z.

Lemma 5.1 shows that there exist p_s^n, q_s^n, x_s^n, $\tilde{z}_s^n$, and z_s^n such that p_s^n, q_s^n, x_s^n, and $\tilde{z}_s^n$ are equilibria of the fictitious economy defined by z_s^n, $\mu_{s\sigma}^n(z^n)$, and $\pi_s^n(z^n)$. By construction, such an equilibrium is consistent and ergodic. Moreover, by (5.3) and the fact that z_s^n is socially feasible, if we let $\alpha^n(s) \subset S'$ be points $\sigma = (y_\sigma, \eta_\sigma)$ with $|y_\sigma^\alpha - z_s^n| \leqslant \alpha$,

$$\lim_{n \to \infty} \sum_{\sigma \in \alpha^n(s)} \pi_{s\sigma}^n(z^n) \to 1. \tag{5.4}$$

Now let $n \to \infty$. Then there is a subsequence such that

$$(p_s^n, q_s^n, x_s^n, \tilde{z}_s^n, \mu_{s\sigma}^n, \pi_s^n) \to (p_s, q_s, x_s, \tilde{z}_s, \mu_{s\sigma}, \pi_s')$$
$$\pi_\sigma^n \mu_{s\sigma}^n / \pi_{s\sigma}^n(z^n) \to \pi_{s\sigma}.$$

By Lemma 5.2 there is a subset $S \subset S'$ such that p_s, q_s, x_s, and $\tilde{z}_s$ restricted to S are consistent ergodic equilibria of the fictitious economy defined by z_s, $\mu_{s\sigma}$, and $\pi_s = \pi_s' / \sum_{\sigma \in S} \pi_\sigma'$ restricted to S. Moreover, by (5.4), $|y_\sigma^a - z_s^a| \leqslant \alpha$ whenever $\mu_{s\sigma} > 0$. By (3.8), this implies for all s, σ

$$|y_\sigma^a - z_s^a| \leqslant \alpha + 2\gamma |\bar{y}| = 2\alpha.$$

Since $\alpha = \varepsilon[1 + \max_{\eta_t \in I} R(\eta_t)]^{-1}/2$ and $r_s = q_s + p_s R(\eta_s)$ with p_s, q_s on the simplex, (5.2) follows.

Finally, we have to show (5.1) holds with B independent of ε. Since in equilibrium agent a's consumption must be socially feasible, and utility is monotone,

$$\sum_{s \in S} \pi_s v^a(p_s, r_s y_s^a, \eta_s, \lambda_s^a) \leqslant \max_{\eta_t \in I} u^a(\bar{x}(\eta_t) + R(\eta_t)\,\bar{y}, \eta_t) \equiv \bar{u}^a. \tag{5.5}$$

Since v^a is Lipshitz with constant λ_s^a, and $v^a(p_s, -p_s \bar{x}^a(\eta_s), \eta_s, \lambda_s^a) = u^a(0, \eta_s) \geqslant \min_{\eta_t \in I} u^a(0, \eta_t) \equiv \underline{u}^a$

$$\sum_{s \in S} \pi_s \lambda_s^a (p_s \bar{x}^a(\eta_s) + r_s \bar{y}_s^a) \leqslant \bar{u}^a - u^a. \tag{5.6}$$

Aggregating over agents yields

$$\sum_{a=1}^{A} \sum_{s \in S} \pi_s \min_a \lambda_s^a [p_s \bar{x}(\eta_s) + r_s \bar{y}] \leqslant \sum_{a=1}^{A} \bar{u}^a - \underline{u}^a. \tag{5.7}$$

Since $\bar{x}(\eta_s)$, $\bar{y}$ are strictly positive, $r_s = q_s + p_s R(\eta_s)$ with $R(\eta_s)$ strictly positive and (p_s, q_s) on the unit simplex, it follows that if we define b to be the minimum of $p_s \bar{x}(\eta_s) + r_s \bar{y}$ over s and (p_s, q_s) on the unit simplex, then $b > 0$. From (5.7), we have

$$b \sum_{a=1}^{A} \left[\sum_{s \in S} \pi_s \min_a \lambda_s^a \right] \leqslant \sum_{a=1}^{A} \bar{u}^a - \underline{u}^a. \tag{5.8}$$

Lemma 5.3 then directly implies (5.1). ∎

LEMMA 5.1. *For fixed s, let* $\mu_{s\sigma}(\cdot)$ *and* $\pi_s(\cdot)$ *be strictly positive continuous functions on* $\mathbb{R}_+^{NAS}$, *satisfying* $\sum_{s \in S} \pi_s(z) = 1$ *and* $\sum_{s \in S} \mu_{s\sigma}(z) = 1$, *for each z. Then there exist* p_s, q_s, x_s, $\tilde{z}_s$, *and* z_s, *such that* p_s, q_s, x_s, *and* $\tilde{z}_s$ *are equilibria of the fictitious economy defined by* z_s, $\mu_{s\sigma}(z)$, *and* $\pi_s(z)$. ∎

Proof. This is a variant on standard finite dimensional general equilibrium theory with the side constraints (3.7). The fact that the demand for assets is a correspondence rather than a function complicates the argument slightly.

Let $P \subset \mathbb{R}_+^{(C+N)S}$ be the space of prices such that each component (p_s, q_s) is on the unit simplex. Let $Z \subset \mathbb{R}_+^{NAS}$ be the space of socially feasible portfolios. We can then define the individual demand set of $(x^a, \tilde{z}^a)$ for $(p, q) \in P$ and initial $z \in Z$ to be the maximizers of (3.5) subject to (3.6), and the additional constraint that x_s^a and z_s^a cannot exceed twice the social total. Because endowments are strictly positive by (A.2) and $\gamma > 0$, the plan $(x^a, \tilde{z}^a) = 0$ strictly satisfies all the budget constraints, implying that budget constraints are lower hemi-continuous. Since $\mu_{\sigma s}$ and π_s are continuous, it follows that demand is upper hemi-continuous (on $P \times Z$), compact, and convex valued. We let $\tilde{Z}(p, q, z) \in \mathbb{R}^{NAS}$ be the demand for assets. Monotonicity of preferences, nonnegative returns (A.4), and the trimming of individual demand imply that as the price of any good or asset goes to zero, the demand for that good or asset equals twice the social total of that, not the good or asset. Aggregate excess demand for goods, $\chi(p, q, z)$, and assets, $\zeta(p, q, z)$, is therefore also upper hemi-continuous, convex, and compact valued, and if $p_{sc} = 0$, $\chi_{sc}(p, q, z) > 0$; if $q_{sj} = 0$, $\zeta_{sj}(p, q, z) > 0$.

Consider the correspondence on $P \times Z$ defined by

$$d(p, q, z) \equiv (p + \chi(p, q, z), q + \zeta(p, q, z), \tilde{Z}(p, q, z)).$$

This is upper hemi-continuous, convex, and compact valued, but does not map into $P \times Z$. However, if $z \in Z$ and (p, q) is equal to the orthogonal projection of $(p + \chi, q + \zeta)$ onto P, then we will indeed have an equilibrium of the type described in the lemma. This method is discussed in Kehoe [3].

There is a large compact convex set $D \supset P \times Z$ such that $d \subset D$. Extend d from $P \times Z$ to D by first projecting onto $P \times Z$, then applying d. By the Kaketani fixed point theorem, this map does have a fixed point. Let (p, q)

be the projection of this fixed point on P. By the boundary conditions, excess demand vanishes at such a point. In particular, $\zeta(p, q, z)=0$. This implies $z \in Z$. ∎

LEMMA 5.2. *Suppose p_s^n, q_s^n, x_s^n, $\tilde{z}_s^n$ are consistent ergodic equilibria of the fictitious economy defined by z_s^n, $\mu_{s\sigma}^n$, and π_s^n, that*

$$(p_s^n, q_s^n, x_s^n, \tilde{z}_s^n, \mu_{s\sigma}^n, \pi_s^n) \to (p_s, q_s, x_s, \tilde{z}_s, z_s, \mu_{s\sigma}, \pi_s),$$

and that $\pi_\sigma^n \mu_{s\sigma}^n / \pi_s^n \to \pi_{s\sigma}$. Then there is a subset $S' \subset S$ such that p_s, q_s, x_s, and $\tilde{z}_s$ restricted to S' are consistent ergodic equilibrium of the economy defined by z_s, $\mu_{s\sigma}$, and $\pi_s / \sum_{\sigma \in S'} \pi_\sigma$ restricted to S'.

Proof. Consider the limit on S (not S'). Clearly $(p_s, q_s, x_s, \tilde{z}_s, z_s, \mu_{s\sigma}, \pi_s)$ satisfy individual budget constraints and social feasibility for each $s \in S$. We further claim that x_s^a, $\tilde{z}_s^a$ are optimal for each a. If not, some $\hat{x}_s^a$, $\hat{z}_s^a$ does strictly better, and since $x_s^a = 0$, $\tilde{z}_s^a = 0$ for all s strictly satisfies the budget constraints, and we may assume $\hat{x}_s^a$, $\hat{z}_s^a$ is strictly feasible (by averaging it with zero). But then for large enough n we will still have $\hat{x}_s^a$, $\hat{z}_s^a$ strictly feasible and strictly better than x_s^{an}, $\tilde{z}_s^{an}$, a contradiction.

Next consider the states S_0 for which $\pi_s = 0$ and S_+ for which $\pi_s > 0$. From (3.9) and $\sum_{\sigma \in S} \pi_{s\sigma}^n = 1$, it follows that since $\pi_s^n > 0$, $\sum_{\sigma \in S} \pi_\sigma^n \mu_{s\sigma} = \pi_s^n$. This implies $\sum_{\sigma \in S} \pi_\sigma \mu_{s\sigma} = \pi_s$. If $s \in S_0$, $\sigma \in S_+$, then $\mu_{s\sigma} = 0$. Consequently, $\hat{x}_s^a$, $\hat{z}_s^a$ satisfies the budget constraints (3.6) only if this is true for the corresponding plan with $(\bar{x}_s^a, \bar{z}_s^a) = (\hat{x}_s^a, \hat{z}_s^a)$ for $s \in S_+$, $(\bar{x}_s^a, \bar{z}_s^a) = 0$ for $s \in S_0$. Conversely, if $\bar{x}_s^a$, $\bar{z}_s^a$ satisfies the budget constraints for $s \in S_+$, $(\hat{x}_s^a, \hat{z}_s^a) = 0$ for $s \in S_0$ satisfies all the budget constraints. Consequently $(p_s, q_s, x_s, \tilde{z}_s, z_s, \mu_{s\sigma}, \pi_s)$ is an equilibrium on S_+.

Finally if S' is an irreducible subclass of S_+, $\mu_{s\sigma} = 0$ if $s \in S_+/S'$ and $\sigma \in S_+$, implying the budget constraints are independent between different irreducible subclasses. Since the objective is additively separable between S' and S_+/S', this implies that $(p_s, q_s, x_s, \tilde{z}_s, z_s, \mu_{s\sigma}, \pi_s / \sum_{s \in S'} \pi_s)$ is an equilibrium. ∎

LEMMA 5.3. *There is a constant λ^* depending only on u such that $\lambda_s^a / \lambda_s^b \leqslant \lambda^*$ in any ergodic equilibrium of any fictitious economy.*

Proof. Define

$$\phi^* = \max_{a, \eta, \bar{x}(\eta) \geqslant x_s^a \geqslant 0} Du^a(x_s^a, \eta) / \min_{a, \eta, \bar{x}(\eta) \geqslant x_s^a \geqslant 0} Du^a(x_s^a, \eta)$$

and fix a. Note that (A.3) ensures that $0 < \phi^* < \infty$. We observe that it is sufficient to prove that $\phi_s^a / \phi_s^b \leqslant \phi^*$ and $1 \leqslant \lambda_s^a / \phi_s^a \leqslant \phi^*$, in which case $\lambda_s^a / \lambda_s^b \leqslant \lambda^* = (\phi^*)^2$. The latter fact follows from (3.12) defining λ_s^a and the first order condition (4.10). Either $J_s^a = -p_s \bar{x}^a(\eta_s)$ and no consumption takes place, so $\lambda_s^a = \phi_s^a$, or $J_s^a > -p_s \bar{x}^a(\eta_s)$ in which case $\lambda_s^a \geqslant \phi_s^a$ and for some c, $\lambda_s^a = p_{sc} \mathrm{D}_c u^a(0, \eta_s)$ and $\phi_s^a = p_{sc} \mathrm{D}_c u^a(x_s^a, \eta_s)$ for some $0 \leqslant x_s^a \leqslant \bar{x}^a(\eta)$. But then $\lambda_s^a / \phi_s^a \leqslant \phi^*$.

We focus then on showing $\phi_s^a / \phi_s^b \leqslant \phi^*$. Observe from (4.10), that for all b

$$\begin{aligned} &\phi_s^b p_s \geqslant \mathrm{D}u^b(x_s^b, n_s) \\ &\phi_s^b q_s \geqslant \delta \sum_{\sigma \in S} (\pi_\sigma \mu_{\sigma s} / \pi_s) \, \phi_\sigma^b r_\sigma \end{aligned} \tag{5.9}$$

with complementary slackness. Moreover, since $r_\sigma > 0$, we must have $q_s > 0$, and we may define

$$B_{s\sigma}^j \equiv \delta(\pi_\sigma \mu_{\sigma s} / \pi_s) \, r_\sigma^j / q_s^j. \tag{5.10}$$

Define the set of states for which a is willing to save by

$$\Sigma \equiv \left\{ s \,\middle|\, \text{for some } j,\ \phi_s^a q_s^j = \delta \sum_{\sigma \in S} (\pi_\sigma \mu_{\sigma s} / \pi_s) \phi_\sigma^a r_\sigma^j \right\}, \tag{5.11}$$

and if $s \in \Sigma$, let $j(s)$ be such a j. If $s \notin \Sigma$, a must be consuming at least his endowment of some good i, so

$$\begin{aligned} &\phi_s^a p_{sc} = \mathrm{D}_c u^a(x_s^a, \eta_s) \\ &\phi_s^b p_{sc} = \mathrm{D}_c u^b(x_s^b, \eta_s) \end{aligned}$$

implying $\phi_s^a / \phi_s^b \leqslant \phi^*$.

Now set $B_{s\sigma} = B_{s\sigma}^{j(s)}$ for $s \in \Sigma$, and let B_Σ be the corresponding square matrix with $\sigma \in \Sigma$. Similarly, define $B_{-\Sigma}$ for $\sigma \notin S$, and partition $\phi^b = (\phi_\Sigma^b, \phi_{-\Sigma}^b)$ in the corresponding way. Notice that $B_\Sigma, B_{-\Sigma}$ are nonnegative matrices with a strictly positive entry in each row.

Suppose first $\Sigma \neq S$. Then

$$\begin{aligned} &\phi_\Sigma^a = B_\Sigma \phi_\Sigma^a + B_{-\Sigma} \phi_{-\Sigma}^a \\ &\phi_\Sigma^b \geqslant B_\Sigma \phi_\Sigma^b + B_{-\Sigma} \phi_{-\Sigma}^b. \end{aligned}$$

Moreover, by (3.9) $\lambda^a \gg 0$. Consequently $(I - B_\Sigma)\phi_\Sigma^a \gg 0$, implying $I - B_\Sigma$ has dominant diagonal, and thus a nonnegative inverse. Consequently,

$$\begin{aligned} &\phi_\Sigma^a = (I - B_\Sigma)^{-1} B_{-\Sigma} \phi_{-\Sigma}^a \\ &\phi_\Sigma^b \geqslant (I - B_\Sigma)^{-1} B_{-\Sigma} \phi_{-\Sigma}^b. \end{aligned}$$

We already showed $\phi_{-\Sigma}^a \leqslant \phi^* \phi_{-\Sigma}^b$, so

$$\phi_\Sigma^a \leqslant \phi^* (I - B_\Sigma)^{-1} B_{-\Sigma} \phi_{-\Sigma}^b \leqslant \phi^* \phi_\Sigma^b.$$

Finally, suppose that $\Sigma = S$. Then, since S is an irreducible class, B is indecomposable. Since $\phi^a = B\phi^a$, B has Froebinius root 1. Consequently, there is a strictly positive vector h such that $h' = h'B$. Then if $\phi^b \neq B\phi^b$, since $\phi_b \geqslant B\phi_b$ are nonnegative vectors, $h'\phi_b > h'B\phi_b$ implying $h'\phi^b > h'\phi^b$, a contradiction. So $\phi^b = B\phi^b$, implying by the Froebinius Theorem $\phi^a = \alpha\phi^b$ for some $\alpha > 0$. But a must consume some good c at some state σ, and for this c and σ, $\phi^a_\sigma p_{\sigma c} = \mathrm{D}_c u^a(x^a_\sigma, \eta_\sigma)$, while $\phi^b_\sigma p_{\sigma c} \geqslant \mathrm{D}_c u^b(x^b_\sigma, \eta_\sigma)$, so that $\alpha \leqslant \phi^*$. ∎

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